PGM - Homework 1

Question 1: Consider a cylindrical water well with a radius of 1 meter. We drop stones from the top-centre of the well. But due to air flow inside the well the stones will not exactly land on the centre at the bottom. The bottom of the well is a circular region represented as

$$C = \{(x, y) \,|\, x^2 + y^2 \leq 1\}. \tag{1}$$

Let the random variables X and Y represent the coordinates of where the stone lands, characterized by the probabilistic density function $f\colon C \mapsto \mathbb{R}$ with the following form:

$$f_{X,Y}(x, y) = \alpha \, \exp(x^2 + y^2) \tag{2}$$

1. What is the value of α? (Hint: You may use polar coordinates.)
2. What is the probability of a stone landing in the region

$$S = \{(x, y) \,|\, \sqrt{x^2 + y^2} \leq \frac{1}{3}\}. \tag{3}$$

3. What is the probability of a stone landing inside

$$S = \{(x, y) \,|\, \frac{1}{3} \leq \sqrt{x^2 + y^2} \leq \frac{2}{3}\}. \tag{4}$$

Question 2: Consider the following joint probability mass function (PMF) $f_{I,N}\colon \{1,2,3,\ldots\} \times \{0,1,2,3,\ldots\} \to [0,1]$:

$$f_{I,N}(i,n) = P(I=i, N=n) = \frac{\alpha}{i^2\, n!} \tag{5}$$

where $i \in \{1,2,3,\cdots\}$ and $n \in= \{0,1,2,\cdots\}$.

1. Derive the value of α?

   Some of the well-known series listed here might help:
   `https://en.wikipedia.org/wiki/Series_(mathematics)`

2. Obtain the formulae for the marginal distributions $f_I(i)$ and $f_N(n)$.

3. Obtain the formulae for the conditional distributions $f_I(i \mid N = n)$ and $f_N(n \mid I = i)$.

4. Are the random variables I and N independent? Why?

Question 3: Consider the following joint PMF $f_{M,N}\colon \{0,1\} \times \{1,2,3\}$:

m	n	$f_{M,N}(m,n)$
0	1	0.10
0	2	0.18
0	3	0.12
1	1	0.14
1	2	0.28
1	3	0.18

1. Derive $f_M(m)$, $f_N(n)$, $f_M(m \mid N = n)$ and $f_N(n \mid M = m)$.

2. Are M and N independent? Why?

Question 4: Assume that the joint distribution

$$P(a,b,c) = \Pr(A=a, B=b, C=c)$$

can be written as

$$P(a,b,c) = \phi(a,b)\,\psi(a,c),$$

where $\phi(a,b) > 0$ and $\psi(a,c) > 0$ for all values of a, b, c.

1. Prove that B and C are conditionally independent given A.

Question 5: Consider the random variables A, B, C takeing on values in sets $D_A = \{0, 1\}$, $D_B = \{-1, 0, 1\}$, $D_C = \{1, 2, 3, 4\}$, respectively. Derive the mininum number of *free* parameters to fully derscribe each of the following distributions

- $P(A, B, C)$
- $P(A \mid B)$
- $P(B \mid A)$
- $P(B \mid A)$
- $P(A \mid B, C)$
- $P(C \mid A, B)$
- $P(A, B, C)$ when we know that A, B, and C are mutually independent.
- $P(A, B, C)$ when we know that B is independent of C given A.
- $P(A, B, C)$ when we know that B is independent of C if $A = 0$.
- $P(A, B, C)$ when we know that $\Pr(A{=}a, B{=}x, C{=}y) = \Pr(A{=}a, B{=}y, C{=}x)$.
- $P(A, B, C)$ when we know that A, B, and C are mutually independent, and $\Pr(C{=}x) = \Pr(B{=}x)$ for all $x \in \{1, 2, 3\}$.